A Note on Disjoint Parallelism

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Abstract: We prove determinism of disjoint parallel programs by applying simple results on abstract reduction systems.

In this note we consider disjoint parallel programs as defined in Hoare [1972]. We assume from the reader the knowledge of **while**-programs (see e.g. De Bakker [1980]) and their semantics defined by means of transitions (see Hennessy and Plotkin [1979]). Two **while**-programs S_1 and S_2 are called *disjoint* if none of them can change the variables accessed by the other one, i.e. if

$$change(S_1) \cap var(S_2) = var(S_1) \cap change(S_2) = \emptyset,$$

where $change(S)$ is the set of variables of S which can be modified by it, i.e. to which a value is assigned within S by means of an assignment. Note that disjoint programs are allowed to read the same variables. For example, the programs

$$x := z \text{ and } y := z$$

are disjoint because $change(x := z) = \{x\}$, $var(y := z) = \{y, z\}$ and $var(x := z) = \{x, z\}, change(y := z) = \{y\}$.

On the other hand the programs $x := z$ and $y := x$ are not disjoint because $x \in change(x := z) \cap var(y := x)$.

Disjoint parallel programs are generated by the same clauses as those defining **while**-programs together with the following clause for *parallel composition*:

$$S ::= [S_1 \| \ldots \| S_n]$$

where for $n \geq 1$, $S_1, \ldots, S_n$ are pairwise disjoint **while**-programs. Thus we do not allow nested parallelism, but we allow parallelism to occur within sequential composition, conditional statements and **while**-loops.

Intuitively, a disjoint parallel program of the form $S \equiv [S_1 \| \ldots \| S_n]$ terminates if and only if all of its components $S_1, \ldots, S_n$ terminate; the final state is then the composition of the final states of $S_1, \ldots, S_n$.

Following Hennessy and Plotkin [1979] we define the semantics of disjoint parallel programs in terms of transitions. Intuitively, a disjoint parallel program $[S_1 \| \ldots \| S_n]$ performs a transition if one of its component performs a transition. Formally, we expand the transition system for **while**-programs by the following rule:

$$\frac{< S_i, \sigma > \; \rightarrow \; < T_i, \tau >}{< [S_1 \| \ldots \| S_i \| \ldots \| S_n], \sigma > \; \rightarrow \; < [S_1 \| \ldots \| T_i \| \ldots \| S_n], \tau >}$$

where $1 \leq i \leq n$.

Recall that computations of disjoint parallel programs are defined as those of **while**-programs. For example

$$\begin{array}{ll} & < [x := 1 \| y := 2 \| z := 3], \sigma > \\ \rightarrow & < [E \| y := 2 \| z := 3], \sigma[1/x] > \\ \rightarrow & < [E \| E \| z := 3], \sigma[1/x][2/y] > \\ \rightarrow & < [E \| E \| E], \sigma[1/x][2/y][3/z] > \end{array}$$

is a computation of $[x := 1 \| y := 2 \| z := 3]$ starting in σ.

Here E stands for the empty program and its occurrence denotes termination of the appropriate component. For example, $[E \| y := 2 \| z := 3]$ denotes a parallel program where the first component has terminated. As explained above a parallel program terminates if and only if all its components terminate. Consequently we identify

$$[E \| \ldots \| E] \equiv E.$$

Thus the final configuration in the above computation is the terminating configuration

$$< E, \sigma[1/x][2/y][3/z] > .$$

We define the partial and total correctness semantics $\mathcal{M}$ and $\mathcal{M}_{tot}$ of disjoint parallel programs by putting for a state σ

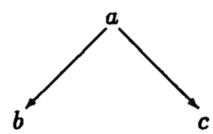
$$\mathcal{M}[[S]](\sigma) = \{\tau \mid < S, \sigma > \rightarrow^* < E, \tau >\}$$

where $\rightarrow^*$ denotes the transitive reflexive closure of $\rightarrow$, and

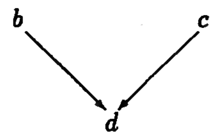
$$\begin{aligned} \mathcal{M}_{tot}[[S]](\sigma) \quad = \quad & \mathcal{M}[[S]](\sigma) \\ \cup \quad & \{\bot \mid S \text{ can diverge from } \sigma\}. \end{aligned}$$

Our aim is to prove that for a disjoint parallel program only one outcome for a given initial state is possible. In other words, for any disjoint parallel program S and state σ, $\mathcal{M}_{tot}[[S]](\sigma)$ has exactly one element. To this end, we need some results concerning abstract reduction systems.

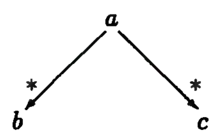
Definition 1 Let $\rightarrow$ be a non-empty binary relation. Denote by $\rightarrow^*$ the transitive reflexive closure of $\rightarrow$. $\rightarrow$ satisfies the *diamond property* if for all a, b, c such that $b \neq c$



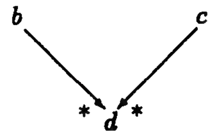
implies that for some d



$\rightarrow$ is called *confluent* if for all a, b, c



implies that for some d



□

The following lemma due to Newman [1942] is of importance to us.

Lemma 2 (Newman's Lemma) If a relation $\rightarrow$ satisfies the diamond property then it is confluent.

Proof. Suppose that $\rightarrow$ satisfies the diamond property. Let $\rightarrow^n$ stand for the n-fold composition of $\rightarrow$. A straightforward proof by induction on $n \geq 0$ shows that $a \rightarrow b$ and $a \rightarrow^n c$ implies that for some $i \leq n$ and some d, $b \rightarrow^i d$ and $c \rightarrow^\epsilon d$. Here $c \rightarrow^\epsilon d$ iff $c \rightarrow d$ or $c = d$. Thus $a \rightarrow b$ and $a \rightarrow^* c$ implies that for some d, $b \rightarrow^* d$ and $c \rightarrow^* d$.

This implies by induction on $n \geq 0$ that if $a \rightarrow^* b$ and $a \rightarrow^n c$ then for some d, $b \rightarrow^* d$ and $c \rightarrow^* d$. This proves confluence. □

We shall also need the following lemma.

Lemma 3 Suppose $\rightarrow$ satisfies the diamond property and that $a \rightarrow b$, $a \rightarrow c$, $b \neq c$. If there exists an infinite sequence $a \rightarrow b \rightarrow \ldots$, then there exists an infinite sequence $a \rightarrow c \rightarrow \ldots$.

Proof. Consider an infinite sequence $a_0 \rightarrow a_1 \rightarrow \ldots$ where $a_0 = a$ and $a_1 = b$.
Case 1. For some $i \geq 0$, $c \rightarrow^* a_i$.

Then $a \rightarrow c \rightarrow^* a_i \rightarrow \ldots$ is the desired sequence.

Case 2. For no $i \geq 0$, $c \rightarrow^* a_i$.

By induction on i we construct an infinite sequence $c_0 \rightarrow c_1 \rightarrow \ldots$ such that $c_0 = c$ and for all $i \geq 0$, $a_i \rightarrow c_i$. For $i = 0$, c_i is already correctly defined.

Consider now the induction step. We have $a_i \rightarrow a_{i+1}$ and $a_i \rightarrow c_i$ for some $i > 0$. Also, since $c \rightarrow^* c_i$, by the assumption $c_i \neq a_{i+1}$. Again by the diamond property for some c_{i+1}, $a_{i+1} \rightarrow c_{i+1}$ and $c_i \rightarrow c_{i+1}$. □

Define now for an element a in the domain of $\rightarrow$

$$\begin{array}{rcl} \mathit{yield}(a) & = & \{b \mid a \rightarrow^* b, b \text{ is } \rightarrow\text{-maximal}\} \\ & \cup & \{\bot \mid \text{ there exists an infinite sequence } a \rightarrow a_1 \rightarrow \ldots\} \end{array}$$

where b is called $\rightarrow$-maximal if for no c, $b \rightarrow c$.

Lemma 4 Suppose that $\rightarrow$ satisfies the diamond property. Then for every a, $yield(a)$ has exactly one element.

Proof. Suppose that for some $\rightarrow$-maximal b and c, $a \rightarrow^* b$ and $a \rightarrow^* c$. By Newman's Lemma for some d, $b \rightarrow^* d$ and $c \rightarrow^* d$. By the $\rightarrow$-maximality of b and c, both $b = d$ and $c = d$, i.e. $b = c$.

Thus the set $\{b \mid a \rightarrow^* b, b \text{ is } \rightarrow\text{-maximal}\}$ has at most one element. If it is empty, then $yield(a) = \{\perp\}$ and we are done.

Otherwise it has exactly one element, say b. Assume by contradiction that there exists an infinite sequence $a \rightarrow a_1 \rightarrow \ldots$. Consider a sequence $b_0 \rightarrow b_1 \rightarrow \ldots \rightarrow b_k$ where $b_0 = a$ and $b_k = b$. Then $k > 0$. Let $b_0 \rightarrow \ldots \rightarrow b_\ell$ be the longest prefix of $b_0 \rightarrow \ldots \rightarrow b_k$ which is an initial fragment of an infinite sequence $a \rightarrow c_1 \rightarrow \ldots$. Then ℓ is well defined, $b_\ell = c_\ell$ and $\ell < k$, since b_k is $\rightarrow$-maximal. Thus $b_\ell \rightarrow b_{\ell+1}$ and $b_\ell \rightarrow c_{\ell+1}$. By the definition of ℓ, $b_{\ell+1} \neq c_{\ell+1}$. By Lemma 3 there exists an infinite sequence $b_\ell \rightarrow b_{\ell+1} \rightarrow \ldots$. This contradicts the choice of ℓ. Thus $yield(a) = \{b\}$. □

We now wish to apply Lemma 4 to the case of disjoint parallel programs. To this purpose we prove first the following lemma.

Lemma 5 (Diamond Property). Let S be a disjoint parallel program and σ a state. Whenever

$$\begin{array}{c} < S, \sigma > \\ \swarrow \quad \searrow \\ < S_1, \sigma_1 > \neq < S_2, \sigma_2 >, \end{array}$$

then for some configuration $< T, \tau >$

$$\begin{array}{c} < S_1, \sigma_1 > \quad < S_2, \sigma_2 > \\ \searrow \quad \swarrow \\ < T, \tau >. \end{array}$$

Proof. By the format of the transition rules, S is of the form $[T_1 \| \ldots \| T_n]$ for some pairwise disjoint **while**-programs $T_1, \ldots, T_n$ and there exist **while**-programs T_i' and T_j', with $i \neq j$, $1 \leq i, j \leq n$ such that

$$\begin{array}{l} S_1 = [T_1 \| \ldots \| T_i' \| \ldots \| T_n], \\ S_2 = [T_1 \| \ldots \| T_j' \| \ldots \| T_n], \\ < T_i, \sigma > \; \rightarrow \; < T_i', \sigma_1 >, \\ < T_j, \sigma > \; \rightarrow \; < T_j', \sigma_2 > . \end{array}$$

Let

$$T = [T_1' \| \ldots \| T_n']$$

where for $k = 1, \ldots, n$ such that $k \neq i$ and $k \neq j$

$$T'_k = T_k$$

If $\sigma_1 \neq \sigma$ then the transition $< S, \sigma > \ \rightarrow \ < S_1, \sigma_1 >$ consists of executing an assignment statement and then $\sigma_1 = \sigma[d_1/u_1]$ for some variable u_1 and element d_1 of the domain D.

Similarly if $\sigma_2 \neq \sigma$ then $\sigma_2 = \sigma[d_2/u_2]$ for some variable u_2 and element d_2 of the underlying domain over which all variables range. Here $\sigma[d/u]$ stand for the state differing from σ only on the variable u to which it assigns the value d.

We now define τ depending on the cardinality of the set $\{\sigma, \sigma_1, \sigma_2\}$:

$$\tau = \begin{cases} \sigma & \text{if } \ card\{\sigma, \sigma_1, \sigma_2\} = 1, \\ \rho & \text{if } \ card\{\sigma, \sigma_1, \sigma_2\} = 2 \text{ and} \\ & \quad \{\sigma, \sigma_1, \sigma_2\} = \{\sigma, \rho\}, \\ \sigma[d_1/u_1][d_2/u_2] & \text{if } \ card\{\sigma, \sigma_1, \sigma_2\} = 3. \end{cases}$$

If $\tau = \sigma[d_1/u_1][d_2/u_2]$ then $< S_1, \sigma_1 > \ \rightarrow \ < T, \tau >$. But by the disjointness condition $\tau = \sigma[d_2/u_2][d_1/u_1]$ which proves that also $< S_2, \sigma_2 > \ \rightarrow \ < T, \tau >$.

Other cases are straightward and left to Jaco. □

As an immediate corollary we obtain the desired result.

Theorem 6 (Determinism) For every disjoint parallel program and a state σ, $\mathcal{M}_{tot}[[S]](\sigma)$ has exactly one element.

Proof. By Lemma's 4 and 5. □

References

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